Intro Video: Section 3.8
Exponential Growth and Decay

Math F251X: Calculus I

IMPORTANT SITUATION:

the rate of change of a function is proportional to the function

$$\frac{df}{dx} = kf \qquad \frac{dy}{dt} = ky$$

We know a function that behaves like this!

$$\frac{d}{dt}\left(\underbrace{e^{kt}}_{y}\right) = k \cdot \underbrace{e^{kt}}_{y}$$

FACT: the only solutions to the differential equation $\frac{dy}{dt} = ky$ are $y = A_0 e^{kt}$ (where A_0 is some constant)

Examples:

Population: constant relative growth rate

$$\frac{dP}{dt} = k\,P \longrightarrow P(t) = P_0\, e^{kt}$$

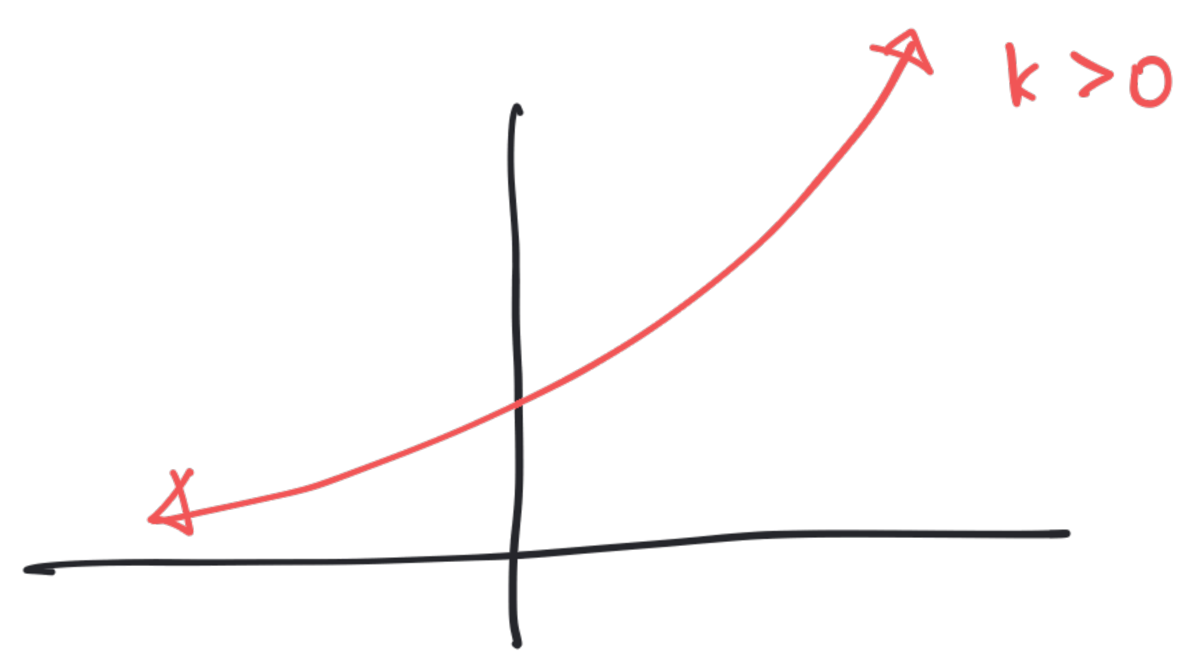
where $P_0 = P(0) = P_0\, e^{k(0)}$

Exponential decay / half-life:

radioactive elements decay at a constant rate

If $k < 0$: decay
$k > 0$: growth

$$\frac{dm}{dt} = k \cdot m$$



Example: Bacteria Growth.

A bacteria culture initially contains 10 cells and grows at a rate proportional to its size. After an hour the population has increased to 400 cells.

a) Find a function $f(t)$ that gives the population after t hours.

$f(t) = P_0 e^{kt}$. Know $f(0) = 10 = P_0 e^{k \cdot 0} \Rightarrow P_0 = 10$.

$f(1) = 400 \Rightarrow 400 = 10 e^{k \cdot 1} \Rightarrow e^k = \frac{400}{10} = 40$

$\Rightarrow \ln(e^k) = \ln(40)$

$\Rightarrow k = \ln(40)$.

$$\boxed{f(t) = 10 e^{\ln(40) \cdot t}}$$

Example: Bacteria Growth.

A bacteria culture initially contains 10 cells and grows at a rate proportional to its size. After an hour the population has increased to 400 cells.

$$f(t) = 10e^{\ln(40)t} = 10\left(e^{\ln(40)}\right)^t = 10 \cdot 40^t$$

- How many bacteria are there after 3 hours?

$f(3) = 10 \cdot 40^3 = 640,000$ bacteria

- How fast is the # of bacteria increasing after 3 hours?

$f'(3)$? Well, $f'(t) = 10 \cdot e^{\ln(40) \cdot t} \cdot \ln(40)$.

So $f'(3) = 10 \cdot 40^3 \cdot \ln(40) = 2,360,882$ bacteria/hour

- How long until there are 10,000,000 bacteria?

$f(t) = 10,000,000 \Rightarrow 10 \cdot 40^t = 10,000,000 \Rightarrow 40^t = 1000000$

$t = \log_{40}(1,000,000) = \dfrac{\ln(1000000)}{\ln(40)} = 3.74518$ hours

Example : Half-life

The half-life of Cesium-137 is 30 years. The 1986 Chernobyl explosion sent about 1000 kg of Cesium-137 into the atmosphere.

1) Find a formula for the amount $m(t)$ of Cesium-137 in the atmosphere after t years (measured since 1986).

$$\left.\begin{aligned} m(0) &= 1000 \\ m(30) &= 500 \end{aligned}\right\} \quad \begin{aligned} m(t) &= 1000\, e^{kt} \\ m(30) &= 1000\, e^{30k} = 500 \end{aligned}$$

So $\dfrac{500}{1000} = e^{30k} \Rightarrow \ln(1/2) = 30k \Rightarrow k = \dfrac{\ln(1/2)}{30}$.

$$m(t) = 1000\, e^{(\ln(1/2)/30)t} = 1000\left(\tfrac{1}{2}\right)^{1/30\, t}$$

2) It is safe for human habitation when less than 100 kg remain. When will this be? $\Rightarrow 100 = 1000\,(1/2)^{t/30} \Rightarrow \dfrac{1}{10} = (1/2)^{t/30}$

$$\ln(1/10) = \ln\left((1/2)^{t/30}\right) = \frac{t}{30}\left(\ln(1/2)\right) \Rightarrow \frac{t}{30} = \frac{\ln(1/10)}{\ln(1/2)}$$

$$\Rightarrow t = \frac{30 \ln(1/10)}{\ln(1/2)} = 99.6578\ y$$

Example: Carbon-14, ^{14}C, has a half-life of about 5730 years. If a piece of ancient parchment has about 74% of the ^{14}C as things alive today, how old is the parchment?

$$Q_0 e^{kt} = m(t) \implies m(5730) = \frac{Q_0}{2} = Q_0 e^{k \cdot 5730} \implies$$

$$\frac{1}{2} = e^{k \cdot 5730} \implies 5730k = \ln(1/2) \implies k = \frac{\ln(1/2)}{5730}$$

Let C = amount of ^{14}C in the parchment originally.

→ What is t when $m(t) = 0.74C$?

$$0.74C = Ce^{(\ln(1/2)/5730)t} \implies 0.74 = e^{\ln(1/2)/5730 \cdot t}$$

$$\implies \ln(.74) = \frac{\ln(1/2)}{5730} t \implies t = \frac{\ln(0.74)(5730)}{\ln(1/2)} \approx 2489 \text{ years}$$

That is, it took 2489 years for the ^{14}C to decay, so the parchment must have left the sheep 2489 years ago.